

EXCITATION OF NONLINEAR ONE-DIMENSIONAL WAKE WAVES IN UNDERDENSE AND OVERDENSE MAGNETIZED PLASMA BY A RELATIVISTIC ELECTRON BUNCH

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The excitation of wake waves by a relativistic homogeneous electron bunch passing through cold magnetized plasma at equilibrium is studied for arbitrary values of the ratio of bunch density to plasma density. The analysis is based on the assumption that the magnetic field is sufficiently strong ($\omega_B \gg \omega_p$, where ω_p and ω_B are the electron plasma and cyclotron frequencies respectively). The periodic and nonperiodic solutions for the momentum of plasma electrons inside the bunch are analyzed. It was shown that the presence of strong external magnetic field may increase the amplitude of wake waves and the maximum transformer ratio, the latter being reached at densities of bunch power than that in the absence of magnetic field. The optimum conditions for obtaining the maximum values of the wake field amplitude and of the transformer ratio are found.

Keywords: Collective effects; Electromagnetic field calculations;
High power beams; Wake-field accelerators

I. INTRODUCTION

The plasma wake-field accelerator (PWFA), proposed by Chen *et al.*¹ in 1985, and discussed in a slightly different context by Fainberg² as early as 1956, uses electrostatic fields in a plasma wave driven by a relativistic electron beam. Charged particles can then be accelerated in this

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electrostatic wake to ultrahigh energies. The first experimental observation, at the Argonne National Laboratory Advanced Accelerator Test Facility, of acceleration by the PWFA mechanism has been reported in Ref. [3]. The one-dimensional theory of the nonlinear regime of the PWFA developed by Rosenzweig,⁴⁻⁶ Ruth *et al.*,⁷ Amatuni *et al.*^{8,9} and Khachatryan¹⁰ predict certain advantages over the linear regime (for more details see reviews^{11,12}). It has been shown that the transformer ratio, the ratio of the maximum decelerating field inside the driving electron bunch to the maximum accelerating field found in the wake of the driver, is enhanced by driving the plasma waves with an ultrarelativistic electron bunch of density one-half that of the plasma electrons. This interesting method of obtaining large transformer ratios is asserted to be more straightforward than alternatives in the linear regime, and also to lessen the multiple scattering that the accelerating particles undergo in a PWFA, as the plasma used need not be as dense as the linear regime requires. The simplicity of the nonlinear scheme for obtaining high transformer ratios, as well as the inherently high accelerating gradients, also makes this method interesting as a candidate for high energy cosmic-ray-acceleration mechanism.

The high transformer ratios obtained in this nonlinear scheme depend on driving the plasma electron density waves to extremely large amplitudes. In contrast to the linear waves, the maximum energy the electrons reach during the oscillation is many times their rest mass energy. The electron oscillation period is an increasing function of this maximum energy. As the amplitude of the wave grows, the wave steepens dramatically, and the positive excursion in density becomes very large in amplitude and narrow in time.

An important result of the nonlinear theory is the proof that the wave breaking limit is $E_{\max} = (mu_b\omega_p/e)[2(\gamma_b - 1)]^{1/2}$ and it is reached at $n_b/n_0 \lesssim 1/(2 + \gamma_b^{-1})$, where $\omega_p = (4\pi n_0 e^2/m)^{1/2}$ is the plasma frequency of electrons, n_b and n_0 are the densities of the bunch and plasma electrons, $\gamma_b = (1 - \beta_b^2)^{-1/2}$ is the relativistic factor of the bunch, $\beta_b = u_b/c$, u_b is the velocity of the bunch. The Dawson¹³ wave breaking limit is equal to $E_{\max} \approx 2mu_p\omega_p/e$ when $\gamma_b \approx 1$ ($\beta_b \ll 1$).

The one-dimensional relativistic strong waves can be excited in the plasma by wide relativistic bunches of charged particles or intense laser pulses^{11,14} (when $k_p a_0 \gg 1$, where $k_p = \omega_p/u_b$, a_0 are the characteristic transverse sizes of bunches or pulses).

The excitation of nonlinear one-dimensional wake waves in underdense and overdense plasma by a relativistic electron bunch was studied in Ref. [9].

In the present work an analogous problem has been treated in the presence of strong external magnetic field \mathbf{B}_0 . A qualitatively novel effect of dependence of the amplitude of wake wave excited by an one-dimensional bunch on the angle α of magnetic field orientation with respect to the direction of bunch motion was found. When α is increased from zero to $\Delta = \arcsin(1/\gamma_b)$, then wake waves with larger amplitudes as well as larger transformer ratios are obtained. Here, the maximum value of the transformer ratio is attained at lower densities of the bunch in comparison with the case of $\mathbf{B}_0 = 0$.

II. BASIC EQUATIONS

Our treatment will start with equations of cold relativistic nonlinear fluid theory. We assume that the plasma ions form an immobile neutralizing background of density n_0 and define the plasma electron density n , velocity $\mathbf{v} = \beta c$, and the plasma frequency $\omega_p = (4\pi n_0 e^2/m)^{1/2}$. We now write the fluid equations for plasma electrons in the presence of an ultrarelativistic beam with homogeneous density n_b , length d , moving with the velocity $\mathbf{u}_b = \beta_b c$ along the z -axis and external constant magnetic field \mathbf{B}_0 ($B_{0x} = B_0 \sin \alpha$, $B_{0y} = 0$, $B_{0z} = B_0 \cos \alpha$, where α is the angle between \mathbf{B}_0 and \mathbf{u}_b):

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - 4\pi e n \beta - 4\pi e \beta_b n_b, \quad (1)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad (2)$$

$$\nabla \cdot \mathbf{E} = -4\pi e(n - n_0) - 4\pi e n_b, \quad (3)$$

$$\frac{\partial \rho}{\partial t} + c(\beta \nabla) \rho = -\frac{e}{mc} [\mathbf{E} + \beta \times (\mathbf{B} + \mathbf{B}_0)], \quad (4)$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0, \quad (5)$$

where $\rho = \beta/(1 - \beta^2)^{-1/2}$, e is the absolute value of electron charge.

If we assume that the wave motion is a function only of the variable $\xi = z - ut$ (the wave ansatz, with u_b taken as the wave phase velocity) we obtain nonlinear differential equations for the dimensionless momentum and field components:

$$n(\xi) = n_0 \frac{\beta_b \sqrt{1 + \rho^2}}{\beta_b \sqrt{1 + \rho^2} - \rho_z}, \quad (6)$$

$$\frac{dE_{x,y}(\xi)}{d\xi} = \frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2}{\lambda_p} \frac{\rho_{x,y}}{\beta_b \sqrt{1 + \rho^2} - \rho_z}, \quad (7)$$

$$\frac{dE_z(\xi)}{d\xi} = -\frac{mc\omega_p}{e} \frac{1}{\lambda_p} \left(\frac{\rho_z}{\beta_b \sqrt{1 + \rho^2} - \rho_z} + \frac{n_b}{n_0} \right), \quad (8)$$

$$\frac{d\rho_x(\xi)}{d\xi} = \frac{e}{mc^2 \beta_b} E_x(\xi) + \frac{\cos \alpha}{\lambda_B} \frac{\rho_y}{\beta_b \sqrt{1 + \rho^2} - \rho_z}, \quad (9)$$

$$\frac{d\rho_y(\xi)}{d\xi} = \frac{e}{mc^2 \beta_b} E_y(\xi) + \frac{1}{\lambda_B} \frac{\rho_z \sin \alpha - \rho_x \cos \alpha}{\beta_b \sqrt{1 + \rho^2} - \rho_z}, \quad (10)$$

$$\begin{aligned} & (\beta_b \sqrt{1 + \rho^2} - \rho_z) \frac{d\rho_z}{d\xi} \\ &= \frac{e}{mc^2} \left[E_z \sqrt{1 + \rho^2} + \frac{1}{\beta_b} (\rho_x E_x + \rho_y E_y) \right] - \frac{\sin \alpha}{\lambda_B} \rho_y, \end{aligned} \quad (11)$$

where $\gamma_b^{-2} = 1 - \beta_b^2$, $\lambda_p = c/\omega_p$, $\lambda_B = c/\omega_B$, $\omega_B = eB_0/mc$. The components of induced magnetic field are determined by means of relations: $E_x = \beta_b B_y$, $E_y = \beta_b B_x$, $B_z = 0$. Note that as it follows from Eqs. (6)–(11), $\rho_x = \rho_y = E_x = E_y = 0$ in the absence of external magnetic field ($\mathbf{B}_0 = 0$) and Eqs. (8) and (11) pass into the well known equations derived in Refs. [9,15]. So, in an anisotropic magnetized plasma, besides the longitudinal motion, the one-dimensional bunch excites the motion of plasma electrons in the direction normal to that of bunch travel. When $\alpha = 0$ (the magnetic field is directed along \mathbf{u}_b) the magnetic field has no influence on the excitation of wake waves and we arrive at the known equations of motion that describe only the longitudinal motion of electrons.

In what follows we shall assume the external magnetic field to be comparatively strong so that the condition $\omega_B \gg \omega_p$ (or $\lambda_B \ll \lambda_p$) were

observed. Hence we obtain the following restriction on unperturbed concentration of plasma electrons:

$$B_0 > 3 \times 10^{-6} \sqrt{n_0}, \quad (12)$$

where n_0 is measured in cm^{-3} , B_0 in kG. Condition (12) is always observed in the range of parameters $n_0 < 10^{15} \text{ cm}^{-3}$ and $B_0 < 100 \text{ kG}$. In the zeroth-order approximation in λ_B one obtains from Eqs. (7) and (9) $\rho_y = E_y = B_x = 0$, from Eq. (10) $\rho_z \sin \alpha = \rho_x \cos \alpha$ (or $\rho_z = \rho \cos \alpha$, $\rho_x = \rho \sin \alpha$), i.e., the plasma electrons move only in the direction of external magnetic field.

Integrating the system of Eqs. (A4) and (A5) (see Appendix) taking into account the boundary conditions at the bunch fronts $\xi = d$ ($E_x(d) = E_z(d) = 0$) and $\xi = 0$ ($E_x(-0) = E_x(+0)$, $E_z(-0) = E_z(+0)$) one obtains the following expressions for the field components E_x and E_z inside ($0 \leq \xi \leq d$, $n_b \neq 0$) and outside ($\xi < 0$, $n_b = 0$) the bunch:

$$0 \leq \xi \leq d;$$

$$E_z(\xi) = -\frac{mc\omega_p}{e} \frac{1}{1 - \gamma_b^2 \sin^2 \alpha} \left(\Phi(\xi) \cos \alpha + \beta_b^2 \gamma_b^2 \sin^2 \alpha \frac{n_b}{n_0} \frac{d - \xi}{\lambda_p} \right), \quad (13)$$

$$E_x(\xi) = \beta_b B_y(\xi) = \frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2 \sin \alpha}{1 - \gamma_b^2 \sin^2 \alpha} \left(\Phi(\xi) + \cos \alpha \frac{n_b}{n_0} \frac{d - \xi}{\lambda_p} \right). \quad (14)$$

$$\xi < 0;$$

$$E_z(\xi) = -\frac{mc\omega_p}{e} \frac{1}{1 - \gamma_b^2 \sin^2 \alpha} \left(\Phi(\xi) \cos \alpha + \beta_b^2 \gamma_b^2 \sin^2 \alpha \frac{n_b}{n_0} \frac{d}{\lambda_p} \right), \quad (15)$$

$$E_x(\xi) = \beta_b B_y(\xi) = \frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2 \sin \alpha}{1 - \gamma_b^2 \sin^2 \alpha} \left(\Phi(\xi) + \cos \alpha \frac{n_b}{n_0} \frac{d}{\lambda_p} \right), \quad (16)$$

where

$$\Phi(\xi) = \lambda_p \frac{d}{d\xi} \left(\cos \alpha \sqrt{1 + \rho^2(\xi)} - \beta_b \rho(\xi) \right). \quad (17)$$

Note that from the continuity of components of the field and momentum of electrons on the boundaries of the bunch there follows the continuity of function $\Phi(\xi)$ at $\xi = 0$ and $\xi = d$.

Thus, Eqs. (A3) and (13)–(17) determine the wake wave in the magnetized plasma for large value of the external magnetic field. For $\alpha = 0$ we obtain from Eqs. (13)–(17) that $E_x = B_y = 0$, and Eqs. (A3), (13) and (15) coincide with equations given in Refs. [9,15], that describe longitudinal oscillations of the plasma electrons. When α has values different from zero, the nature of solutions of Eqs. (A3) and (15)–(17) is drastically changed. First, the behavior of solution of Eq. (A3) depends on the fact whether the angle α is greater or less than a definite value $\Delta = \arcsin(1/\gamma_b)$. Second, owing to an anisotropy of plasma due to the presence of an external magnetic field, besides the longitudinal components there arise transverse components (as well as an induced magnetic field). Furthermore, owing to the anisotropy of plasma the components of the wake field (Eqs. (13)–(17)) contain the Coulomb field of the bunch (the second terms in Eqs. (13)–(17)) that depends along with other parameters also on the angle α .

From expressions (A3) and (13)–(17) it follows that periodic wake fields behind the bunch arise only at values of angle α satisfying the condition $\alpha < \Delta$. For larger values of α ($\alpha > \Delta$) the momentum of plasma electrons and wake fields monotonically increase with distance from the bunch. Below we shall analyze the Eq. (A3) inside and behind the bunch only for the values $\alpha < \Delta$.

III. ELECTRON MOMENTUM INSIDE THE BUNCH FOR ARBITRARY VALUES OF THE RATIO n_b/n_0

Now let us consider the equation of plasma motion (A3) inside the bunch. In this range $0 \leq \xi \leq d$, $n_b = \text{const} \neq 0$. The integration of Eq. (A3) with due regard for boundary conditions $\rho(d) = \Phi(d) = 0$ will give

$$\Phi(\xi) = \pm \sqrt{2} \left(A - A\sqrt{1 + \rho^2} - B\rho \right)^{1/2}, \quad (18)$$

where

$$A = 1 - \gamma_b^2 \sin^2 \alpha - \frac{n_b}{n_0} \cos^2 \alpha, \quad B = \frac{n_b}{n_0} \beta_b \cos \alpha. \quad (19)$$

In Eq. (18) one is to take the sign plus (minus) when $\rho(\xi)$ decreases (increases) with increasing ξ . As it follows from the Eq. (18), for solution of Eq. (A3) it is necessary that the radicand in (18) assume only the positive values. Hence follows a restriction on the momentum of plasma electrons: $\rho \leq 0$. Taking into account this restriction one is to take the sign “minus” during the integration of Eq. (18). Making allowance for boundary conditions at the front boundary of the bunch we obtain from (17) and (18)

$$\frac{d - \xi}{\lambda_p} \sqrt{2} = \int_{\rho}^0 \frac{(\beta_b \sqrt{1 + \eta^2} - \eta \cos \alpha) d\eta}{\sqrt{1 + \eta^2} \sqrt{A - A\sqrt{1 + \eta^2} - B\eta}}, \quad (20)$$

that determines an implicit dependence of ρ on ξ . Note that the expression obtained in [9] will follow from (20) at $\alpha = 0$.

Consider first the case of small densities of the bunch.

A. The Case of $n_b/n_0 < \gamma_b^2(\cos \alpha - \beta_b)/\cos \alpha = b_1$

In this case $0 < B < A$ and one can easily show that the range of variation of ρ is determined by the relation $-\rho_0 \leq \rho \leq 0$, where

$$\rho_0 = \frac{2\beta_b \varepsilon}{1 - \beta_b^2 \varepsilon^2}, \quad \varepsilon = \frac{n_b/n_0 \cos \alpha}{1 - \gamma_b^2 \sin^2 \alpha - n_b/n_0 \cos^2 \alpha}. \quad (21)$$

Then the dependence of ρ on ξ has a periodic behavior with oscillation amplitude ρ_0 .

The integration of (20) will give an implicit dependence of ρ on ξ for afore-mentioned values of the magnetic field orientation angle α and density of bunch:

$$\begin{aligned} \frac{d - \xi}{\lambda_p} = & \frac{1}{1 - \beta_b \varepsilon} \sqrt{\frac{2(1 + \varepsilon \cos \alpha)}{1 - \gamma_b^2 \sin^2 \alpha}} \left\{ \frac{\sqrt{2}\beta_b(1 + \varepsilon \cos \alpha)}{\sqrt{1 + \beta_b \varepsilon}} E(\psi, k) \right. \\ & \left. - (\beta_b + \cos \alpha) \sqrt{1 - \beta_b \varepsilon \rho - \sqrt{1 + \rho^2}} \right\}, \end{aligned} \quad (22)$$

where

$$\psi = \arcsin\left(\frac{1}{k}\sqrt{1-\rho-\sqrt{1+\rho^2}}\right), \quad k = \sqrt{\frac{2\beta_b\varepsilon}{1+\beta_b\varepsilon}} < 1, \quad (23)$$

$E(\psi, k)$ is the elliptic integral of the second kind, $0 \leq \beta_b\varepsilon < 1$. When $\rho = \rho_0$, assuming in Eq. (22) that $\xi = 0$ one obtains the length of bunch d_0 or the wave half length inside the bunch

$$\lambda_{in} = 2d_0 = \frac{4\lambda_p\beta_b}{1-\beta_b\varepsilon} \sqrt{\frac{(1+\varepsilon\cos\alpha)^3}{(1-\gamma_b^2\sin^2\alpha)(1+\beta_b\varepsilon)}} E(k), \quad (24)$$

where $E(k)$ is a complete elliptic integral of the second kind.

The dependence of wavelength inside the bunch on the bunch density is shown in Figure 1 for values of $\alpha = 0$ (solid line), $\alpha = 0.2\Delta$ (dashed line), $\alpha = 0.4\Delta$ (dotted line) and $0 \leq n_b/n_0 \leq b_1$.

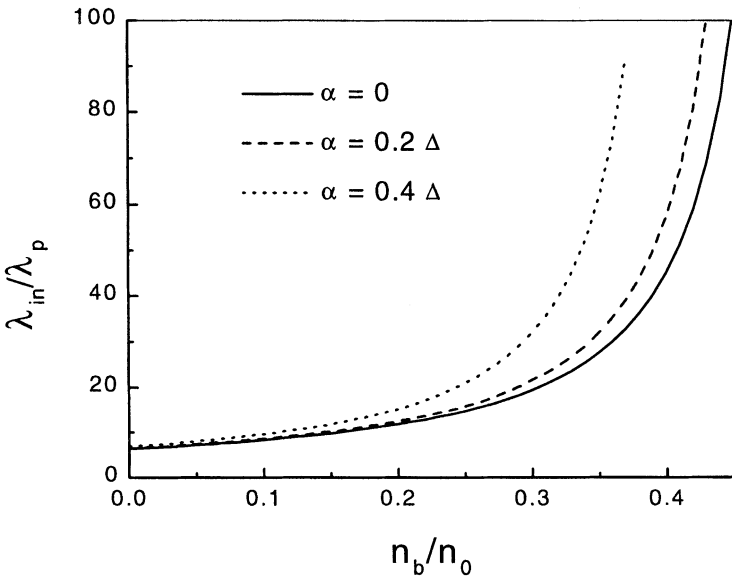


FIGURE 1 The dependence of the λ_{in}/λ_p on n_b/n_0 ($0 \leq n_b/n_0 \leq b_1$) for $\gamma_b = 50$, $\alpha = 0$ (solid line), $\alpha = 0.2\Delta$ (dashed line) and $\alpha = 0.4\Delta$ (dotted line).

For values of ratio $n_b/n_0 \geq b_1$ the only constraint on the region of allowed values of ρ is the condition $\rho \leq 0$. Therefore, the dependence of ρ on ξ will get nonperiodic and $\rho(\xi)$ will increase infinitely with ξ .

B. The Case of $n_b/n_0 = b_1$

In this case $A = B$ and integration of expression (20) gives

$$\begin{aligned} \frac{d - \xi}{\lambda_p} = & \frac{1}{2} \sqrt{\frac{(\cos \alpha + \beta_b)^3}{2\beta_b(1 - \gamma_b^2 \sin^2 \alpha)}} \\ & \times \left\{ \sqrt{1 - \rho - \sqrt{1 + \rho^2}} \left(\sqrt{1 + \rho^2} - \rho - 2 \frac{\cos \alpha - \beta_b}{\cos \alpha + \beta_b} \right) \right. \\ & \left. + \ln \left[\left(1 + \sqrt{1 - \rho - \sqrt{1 + \rho^2}} \right) \sqrt{\sqrt{1 + \rho^2} - \rho} \right] \right\}. \quad (25) \end{aligned}$$

One can see from Eq. (25) that at large values of $d - \xi \gg \lambda_p$, i.e., with the distance from the front boundary of the bunch $|\rho(\xi)|$ grows linearly depending on $d - \xi$.

C. The Case of $b_1 < n_b/n_0 < \gamma_b^2(\cos \alpha + \beta_b)/\cos \alpha = b_2$

In this case $-B < A < B$. Taking into account this inequality one obtains from Eq. (20)

$$\begin{aligned} \frac{d - \xi}{\lambda_p} = & \frac{\sqrt{2}}{B^2 - A^2} \left\{ (B + |A|)(\cos \alpha + \beta_b) \sqrt{A - A\sqrt{1 + \rho^2} - B\rho} \right. \\ & \left. + (B \cos \alpha + \beta_b |A|) \left[\frac{B - |A|}{\sqrt{B}} F(\chi, \kappa) - 2\sqrt{BE}(\chi, \kappa) \right] \right\}, \quad (26) \end{aligned}$$

where

$$\chi = \arccos \sqrt{\sqrt{1 + \rho^2} + \rho}, \quad \kappa = \sqrt{\frac{B + |A|}{2B}}, \quad (27)$$

$F(\chi, \kappa)$ is an elliptic integral of the first kind, $\kappa < 1$.

D. The Case of $n_b/n_0 = b_2$

In this case $A = -B$. One obtains from the general form of Eq. (20)

$$\begin{aligned} \frac{d - \xi}{\lambda_p} = \frac{1}{\sqrt{2B}} \left\{ \left[\cos \alpha + \beta_b - \frac{1}{2}(\cos \alpha - \beta_b) \right. \right. \\ \times \left. \left(\sqrt{1 + \rho^2} + \rho \right) \right] \sqrt{\sqrt{1 + \rho^2} - \rho - 1} \\ \left. - \frac{1}{2}(\cos \alpha - \beta_b) \operatorname{arctg} \sqrt{\sqrt{1 + \rho^2} - \rho - 1} \right\}. \end{aligned} \quad (28)$$

E. The Case of $n_b/n_0 > b_2$

Here $A < -B$. The calculation of the integral in Eq. (20) gives

$$\begin{aligned} \frac{d - \xi}{\lambda_p} = \frac{\sqrt{2}}{A^2 - B^2} \left\{ \sqrt{A - A\sqrt{1 + \rho^2} - B\rho} \right. \\ \times \left[|A| \cos \alpha - \beta_b B + (\beta_b |A| - B \cos \alpha) \right. \\ \times \left. \frac{\sqrt{A^2 - B^2} - A\sqrt{1 + \rho^2} - B\rho}{B\sqrt{1 + \rho^2} + A\rho} \right] \\ + (B \cos \alpha + \beta_b A) \left[\sqrt{\sqrt{A^2 - B^2} - A} E(\sigma, r) \right. \\ \left. + \frac{A}{\sqrt{\sqrt{A^2 - B^2} - A}} F(\sigma, r) \right] \left. \right\}, \end{aligned} \quad (29)$$

where

$$\sigma = \arcsin \sqrt{\frac{A\sqrt{1 + \rho^2} + B\rho - A}{A\sqrt{1 + \rho^2} + B\rho + \sqrt{A^2 - B^2}}}, \quad r = \sqrt{\frac{2\sqrt{A^2 - B^2}}{\sqrt{A^2 - B^2} - A}}. \quad (30)$$

In all cases under consideration in items 3–5, at large values of $d - \xi \gg \lambda_p |\rho(\xi)|$ is a quadratic function of the distance to the front

boundary of the bunch $d - \xi$. At $\alpha = 0$ the expressions obtained in items 1–5 coincide with the results given in Ref. [9].

IV. THE FIELDS INSIDE THE BUNCH FOR ARBITRARY VALUES OF THE RATIO n_b/n_0

In this section the Eqs. (13) and (14) describing the induced electromagnetic fields inside the bunch were analyzed. Consider at first the case $\alpha \neq \Delta$. We shall begin with an analysis of a transverse component of electric field E_x . In the previous section it was shown that the momentum of electrons inside the bunch meets the condition $\rho(\xi) \leq 0$. As follows from this inequality, Eq. (7) and the boundary condition $E_x(d) = 0$, inside the bunch E_x is positive and monotonically increases with distance from the front boundary of the bunch. The function E_x (as well as B_y) takes on the maximum value at the rear boundary of the bunch, at $\xi = 0$. From Eq. (13) one obtains for the maximum value of E_x

$$E_{x\max} = \frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2 \sin \alpha}{1 - \gamma_b^2 \sin^2 \alpha} \times \left(\pm 2^{1/2} \sqrt{A - A\sqrt{1 + \rho^2(0)} - B\rho(0)} + \frac{n_b}{n_0} \frac{d}{\lambda_p} \cos \alpha \right). \quad (31)$$

In Eq. (31) one takes the sign “plus” (“minus”) if $\rho(\xi)$ is decreased (increased) with increasing of ξ . In particular, if $n_b/n_0 < b_1$ (the momentum of electrons inside the bunch is periodically changing with ξ and is given by Eq. (22)) and $d = \lambda_{in}/2$, then $\rho(0) = -\rho_0$ and the maximum value of E_x as obtained from (31) is

$$E_{x\max} = \frac{mc\omega_p}{e} \frac{n_b}{n_0} \frac{\lambda_{in}}{2\lambda_p} \frac{\beta_b^2 \gamma_b^2 \sin \alpha \cos \alpha}{1 - \gamma_b^2 \sin^2 \alpha}. \quad (32)$$

In all other cases discussed in the previous section, $E_{x\max}$ is determined by the Eq. (31) with the sign “minus”.

The longitudinal component E_z of the field satisfies the Eq. (8). In case of $n_b/n_0 \geq \cos \alpha / (\beta_b + \cos \alpha) = b_c$, $dE_z/d\xi < 0$ and $E_z(\xi)$ is positive and monotonically increases with the distance from the front boundary of the bunch. In case of $n_b/n_0 < b_c$ the longitudinal field E_z first monotonically increases with the distance from the front boundary of the bunch, than reaching the maximum at $\rho(\tilde{\xi}) = -\tilde{\rho}$, where

$$\tilde{\rho} = \frac{\beta_b \theta}{\sqrt{\cos^2 \alpha - \beta_b^2 \theta^2}}, \quad \theta = \frac{n_b/n_0}{1 - n_b/n_0}, \quad (33)$$

monotonically decreases to the rear boundary ($\xi = 0$).

The maximum value of longitudinal field E_z in the point $\tilde{\xi}$ follows from expression (13), where the substitutions $\rho = -\tilde{\rho}$ and $\xi = \tilde{\xi}$ are made:

$$\begin{aligned} \tilde{E}_z = & \frac{mc\omega_p/e}{1 - \gamma_b^2 \sin^2 \alpha} \\ & \times \left(2^{1/2} \cos \alpha \sqrt{A - A\sqrt{1 + \tilde{\rho}^2} + B\tilde{\rho} - \beta_b^2 \gamma_b^2 \sin^2 \alpha \frac{n_b}{n_0} \frac{d - \tilde{\xi}}{\lambda_p}} \right), \end{aligned} \quad (34)$$

where the value of $(d - \tilde{\xi})/\lambda_p$ is expressed through $\tilde{\rho}$ by means of expressions (22), (25), (26), (28) and (29). The value of longitudinal field at the rear boundary of the bunch ($\xi = 0$) follows from the Eq. (13):

$$\begin{aligned} E_z(0) = & \frac{-mc\omega_p/e}{1 - \gamma_b^2 \sin^2 \alpha} \\ & \times \left(\pm 2^{1/2} \cos \alpha \sqrt{A - A\sqrt{1 + \rho^2(0)} - B\rho(0) + \beta_b^2 \gamma_b^2 \sin^2 \alpha \frac{n_b}{n_0} \frac{d}{\lambda_p}} \right). \end{aligned} \quad (35)$$

In the general case the maximum value of longitudinal field inside the bunch is

$$E_{z \max} = \max \left[|E_z(0)|; \tilde{E}_z \right]. \quad (36)$$

In particular, if $n_b/n_0 < b_1$ and $d = \lambda_{in}/2$ the value of \tilde{E}_z is determined by the Eq. (36), and for $E_z(0)$ we obtain from (13), (18), (21) and (24):

$$E_z(0) = -\frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2 \sin^2 \alpha}{1 - \gamma_b^2 \sin^2 \alpha} \frac{n_b}{n_0} \frac{\lambda_{in}}{2\lambda_p} < 0. \quad (37)$$

Below we shall consider the equation of motion of plasma (A3) outside the bunch ($\xi < 0$).

V. THE WAKE FIELD

To find the wake field $E_x(\xi)$ and $E_z(\xi)$ behind the bunch ($\xi \leq 0$), one has to integrate Eq. (A3) for $n_b = 0$ taking into account the condition of continuity of fields and momentum ρ at the rear boundary of the bunch $\xi = 0$. In this case the fields are determined by means of expressions (15)–(17), where now

$$\Phi(\rho) = \pm \sqrt{2(1 - \gamma_b^2 \sin^2 \alpha) \left(\sqrt{1 + \rho_{\max}^2} - \sqrt{1 + \rho^2} \right)}. \quad (38)$$

Here

$$\rho_{\max} = \sqrt{\left[\frac{A - B\rho(0)}{1 - \gamma_b^2 \sin^2 \alpha} + \sqrt{1 + \rho^2(0)} \left(1 - \frac{A}{1 - \gamma_b^2 \sin^2 \alpha} \right) \right]^2 - 1} \quad (39)$$

is the maximum allowable value of the momentum $\rho(\xi)$. As is seen from expression (38), the momentum $\rho(\xi)$ (hence, the fields $E_x(\xi)$, $E_z(\xi)$ and function $\Phi(\xi)$) is a periodic function and changes in the range $-\rho_{\max} \leq \rho(\xi) \leq \rho_{\max}$. The wake wavelength is determined from expressions (17) and (38) and has the following form:

$$\lambda_{out} = \frac{4\sqrt{2}\beta_b\lambda_p}{\sqrt{1 - \gamma_b^2 \sin^2 \alpha}} \sqrt{\frac{2}{1 - p^2} \left[E(p) - \frac{1 - p^2}{2} K(p) \right]}, \quad (40)$$

where

$$p = \sqrt{\frac{\sqrt{1 + \rho_{\max}^2} - 1}{\sqrt{1 + \rho_{\max}^2} + 1}}. \quad (41)$$

The dependence of wake wavelength (λ_{out}) on its amplitude ρ_{\max} is shown in Figure 2. It is seen that the wavelength increases with the amplitude and the angle α .

As it follows from Eqs. (15)–(17), (38) and (39) the transverse ($E_x(\xi)$) and longitudinal ($E_z(\xi)$) wake waves change within the ranges $\mathcal{E}_{1x} \leq E_x(\xi) \leq \mathcal{E}_{2x}$, $\mathcal{E}_{1z} \leq E_z(\xi) \leq \mathcal{E}_{2z}$, respectively, where

$$\begin{aligned} \begin{pmatrix} \mathcal{E}_{1x} \\ \mathcal{E}_{2x} \end{pmatrix} &= \frac{mc\omega_p}{e} \frac{\beta_b^2 \gamma_b^2 \sin \alpha}{1 - \gamma_b^2 \sin^2 \alpha} \left[\cos \alpha \frac{n_b}{n_0} \frac{d}{\lambda_p} \right. \\ &\quad \left. \mp \left(\frac{2n_b}{n_0} \cos \alpha \right)^{1/2} \sqrt{\cos \alpha \left(\sqrt{1 + \rho^2(0)} - 1 \right) - \beta_b \rho(0)} \right], \end{aligned} \quad (42)$$

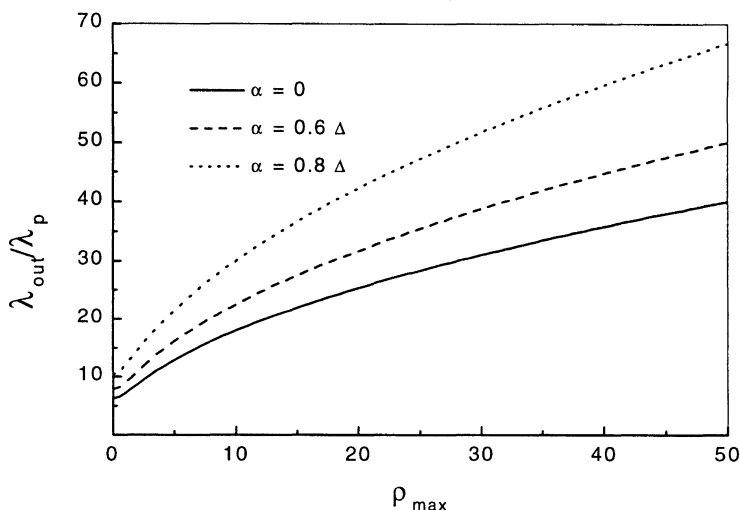


FIGURE 2 The dependence of the $\lambda_{\text{out}}/\lambda_p$ on ρ_{\max} for $\gamma_b=50$, $\alpha=0$ (solid line), $\alpha=0.6\Delta$ (dashed line) and $\alpha=0.8\Delta$ (dotted line).

$$\begin{pmatrix} \mathcal{E}_{1z} \\ \mathcal{E}_{2z} \end{pmatrix} = -\frac{mc\omega_p/e}{1 - \gamma_b^2 \sin^2 \alpha} \left[\beta_b^2 \gamma_b^2 \sin^2 \alpha \frac{n_b}{n_0} \frac{d}{\lambda_p} \pm \left(\frac{2n_b}{n_0} \cos^3 \alpha \right)^{1/2} \right. \\ \left. \times \sqrt{\cos \alpha \left(\sqrt{1 + \rho^2(0)} - 1 \right)} - \beta_b \rho(0) \right]. \quad (43)$$

It is seen from obtained expressions that $|\mathcal{E}_{1x}| < \mathcal{E}_{2x}$ and $|\mathcal{E}_{1z}| > |\mathcal{E}_{2z}|$. Thus, the maximum values of transverse and longitudinal fields behind the bunch are \mathcal{E}_{2x} and $|\mathcal{E}_{1z}|$ respectively. Here the value of transverse field varies with respect to some positive quantity (the first term in Eq. (42)), and the value of longitudinal field varies with respect to some negative quantity (the first term in Eq. (43)).

Now consider the efficiency of wake field excitation for different values of the density and thickness of the bunch. If the momentum of plasma electrons inside the bunch is a periodic function (the bunch density meets the condition of subsection A of Section III, $n_b/n_0 < b_1$), then inside the bunch the variation range of ρ is defined by the relation $-\rho_0 \leq \rho \leq 0$, where ρ_0 is determined by expression (21). In case when an integer number of wavelengths is interposed within the bunch ($d = (n+1)\lambda_{in}$, where $n=0, 1, 2, \dots$), then it follows from results of subsection A of the Section III that $\rho(0) = 0$, and in virtue of expression (39) $\rho_{\max} = 0$. In this case (as in the absence of magnetic field [9]) the momentum of electrons for $\xi \leq 0$ is zero, and the density of plasma is coincident with the equilibrium value n_0 . As follows from expressions (15), (16), (38) and (39), the components of electric fields for $\xi \leq 0$ are constant and are determined by the second terms of expressions (15) and (16). Thus, when the condition of $d = (n+1)\lambda_{in}$ is observed in an anisotropic plasma, a constant electric field may arise behind the bunch that is perpendicular to the external magnetic field according to expressions (42) and (43).

In case of $d = (n+1/2)\lambda_{in}$, the amplitude of wake field oscillations is maximum $\rho_{\max} = |\rho(0)| = \rho_0$. The maximum values of electric field components are

$$E_{z\max}^{(out)} = \frac{mc\omega_p}{e} \beta_b \varepsilon \left[\frac{2 \cos \alpha}{\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)(1 - \beta_b^2 \varepsilon^2)}} + \frac{(n+1/2)\lambda_{in}}{\lambda_p} \frac{\beta_b \gamma_b^2 \sin^2 \alpha}{\cos \alpha (1 + \varepsilon \cos \alpha)} \right], \quad (44)$$

$$E_{x\max}^{(\text{out})} = \frac{mc\omega_p}{e} \varepsilon \beta_b^2 \gamma_b^2 \sin \alpha \left[\frac{2\beta_b}{\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)(1 - \beta_b^2 \varepsilon^2)}} + \frac{\lambda_{\text{in}}}{\lambda_p} \frac{n + 1/2}{1 + \varepsilon \cos \alpha} \right]. \quad (45)$$

The condition of positivity of plasma density $n(\xi)$ (Eq. (6)) imposes a restriction on possible values of the momentum of plasma electrons at $\xi < 0$

$$-\infty < \rho(\xi) \leq \rho_{\max} \leq \frac{\beta_b \gamma_b}{\sqrt{1 - \gamma_b^2 \sin^2 \alpha}}, \quad (46)$$

that, in its turn, limits the values of electron momentum at the rear boundary of the bunch ($\xi = 0$). For example, for the value of bunch density $n_b/n_0 < b_1$ Eq. (46) yields

$$-\rho_{\max}(0) \equiv -\frac{\beta_b \gamma_b}{\sqrt{1 - \gamma_b^2 \sin^2 \alpha}} \leq \rho(0) \leq 0. \quad (47)$$

For a bunch, the density of which satisfies the condition $n_b/n_0 < b_1$, the electron momentum inside the bunch is a periodical function. In this case from expressions (21) and (47) we obtain a restriction on the density of bunch, that is more rigid than the condition $n_b/n_0 < b_1$

$$\frac{n_b}{n_0} \leq \frac{1 - \gamma_b^2 \sin^2 \alpha}{\cos \alpha \left(2 \cos \alpha + \sqrt{\gamma_b^{-2} - \sin^2 \alpha} \right)} = b_0 < b_1. \quad (48)$$

For the density of bunch $n_b/n_0 = b_0$, the maximum values of fields have the following forms:

$$E_{z\max}^{(\text{out})} = \frac{mc\omega_p}{e} \sqrt{\gamma_b^2 - 1} \times \left[\frac{\sqrt{2} \cos \alpha}{\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)^{3/2} (\gamma_b \cos \alpha + \sqrt{1 - \gamma_b^2 \sin^2 \alpha})}} + \frac{\beta_b \gamma_b^2 \sin^2 \alpha (\lambda_{\text{in}}/\lambda_p) (n + 1/2)}{\cos \alpha (2 \cos \alpha + \sqrt{\gamma_b^{-2} - \sin^2 \alpha})} \right], \quad (49)$$

$$E_{x \max}^{(\text{out})} = \frac{mc\omega_p}{e} \beta_b^2 \gamma_b^3 \sin \alpha \times \left[\frac{\sqrt{2}\beta_b}{\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)^{3/2} (\gamma_b \cos \alpha + \sqrt{1 - \gamma_b^2 \sin^2 \alpha})}} + \frac{(\lambda_{\text{in}}/\lambda_p)(n + 1/2)}{2\gamma_b \cos \alpha + \sqrt{1 - \gamma_b^2 \sin^2 \alpha}} \right]. \quad (50)$$

The dependence of wake waves on ξ is shown in Figures 3–6 for the values $\gamma_b = 60$, $d = \lambda_{\text{in}}/2$, $n_b/n_0 = b_0(\alpha)$ (Figures 3 and 4) and $n_b/n_0 = 0.8b_0(\alpha)$ (Figures 5 and 6). Note that in Figure 3 the curve calculated for the value $\alpha = 0$ coincides with the result obtained in Ref. [9].

In case of even higher densities of the bunch $n_b/n_0 > b_0$, there is also a limit on the thickness of the bunch $d \leq d_{\text{max}}$, where d_{max} is determined from expression (20) if in the latter we put $\xi = 0$ and $\rho = -\rho_{\text{max}}(0)$:

$$d_{\text{max}} = \frac{\lambda_p}{\sqrt{2}} \int_0^{\rho_{\text{max}}(0)} \frac{(\beta_b \sqrt{1 + \eta^2} + \eta \cos \alpha) d\eta}{\sqrt{1 + \eta^2} \sqrt{A - A\sqrt{1 + \eta^2} + B\eta}}. \quad (51)$$

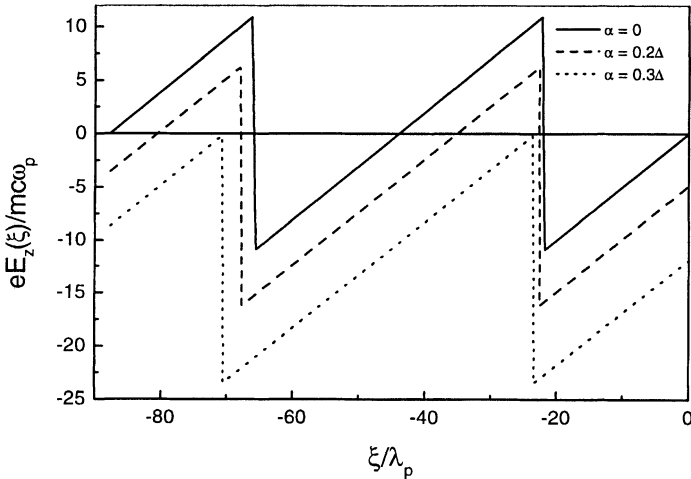


FIGURE 3 The dependence of the function $eE_z(\xi)/mc\omega_p$ on ξ/λ_p for $n_b/n_0 = b_0(\alpha)$, $d = \lambda_{\text{in}}/2$, $\gamma_b = 60$, $\alpha = 0$ (solid line), $\alpha = 0.2\Delta$ (dashed line) and $\alpha = 0.3\Delta$ (dotted line).

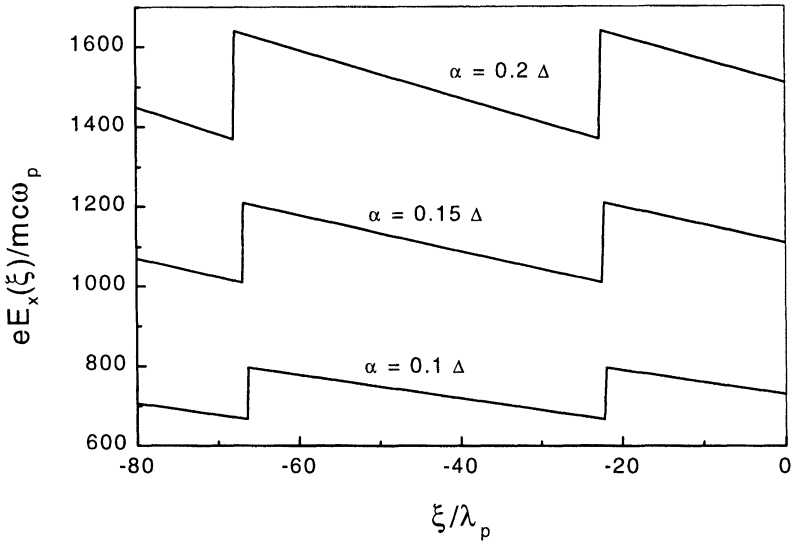


FIGURE 4 The dependence of the function $eE_x(\xi)/mc\omega_p$ on ξ/λ_p for $n_b/n_0 = b_0(\alpha)$, $d = \lambda_{in}/2$, $\gamma_b = 60$, $\alpha = 0.1\Delta$, $\alpha = 0.15\Delta$ and $\alpha = 0.2\Delta$.

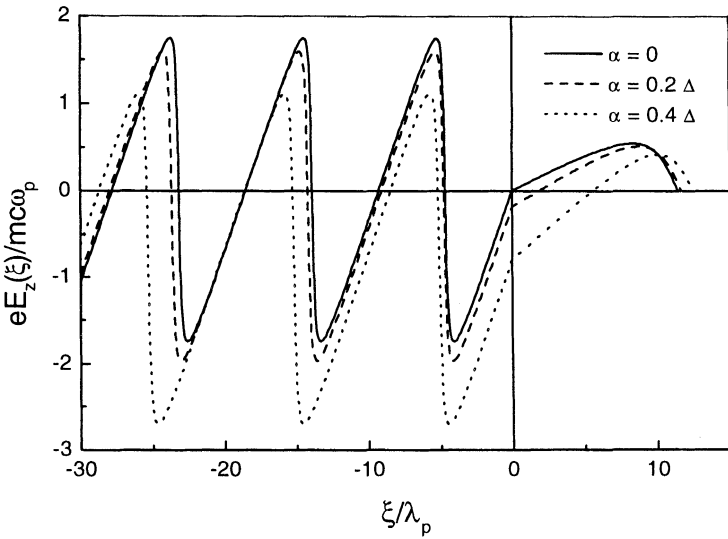


FIGURE 5 The dependence of the function $eE_z(\xi)/mc\omega_p$ on ξ/λ_p for $n_b/n_0 = 0.8b_0(\alpha)$, $d = \lambda_{in}/2$, $\gamma_b = 60$, $\alpha = 0$ (solid line), $\alpha = 0.2\Delta$ (dashed line) and $\alpha = 0.4\Delta$ (dotted line).

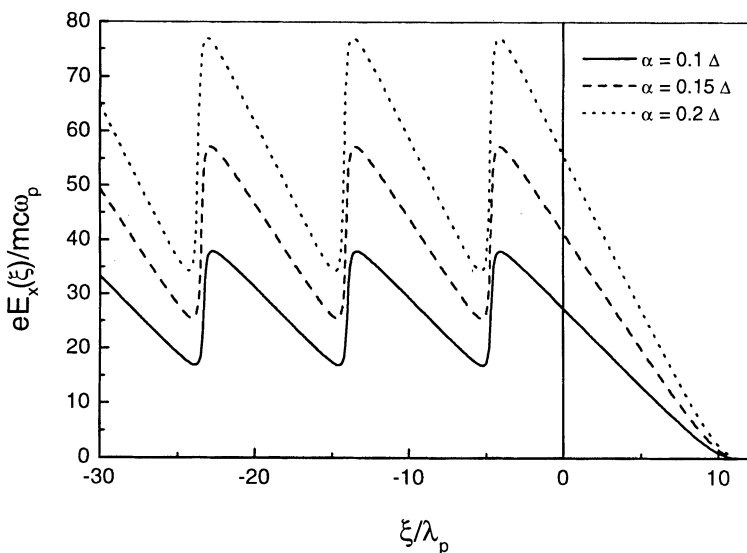


FIGURE 6 The dependence of the function $eE_x(\xi)/mc\omega_p$ on ξ/λ_p for $n_b/n_0 = 0.8b_0(\alpha)$, $d = \lambda_{in}/2$, $\gamma_b = 60$, $\alpha = 0.1\Delta$ (solid line), $\alpha = 0.15\Delta$ (dashed line) and $\alpha = 0.2\Delta$ (dotted line).

When $b_0 < n_b/n_0 < b_1$, the thickness of the bunch should satisfy the condition $0 \leq d \leq d_{\max}$ or $n\lambda_{in} - d_{\max} \leq d \leq n\lambda_{in} + d_{\max}$, where $n = 1, 2, 3, \dots$. Note that in this case $d_{\max} < \lambda_{in}/2$. For $n_b/n_0 \geq b_1$ the thickness of the bunch satisfies only the condition $0 \leq d \leq d_{\max}$.

In Figure 7 the dependence of maximum thickness of the bunch its density is shown at $n_b/n_0 > b_0$ for $\alpha = 0$ (solid line), $\alpha = 0.4\Delta$ (dashed line), $\alpha = 0.8\Delta$ (dotted line). As was expected, d_{\max} decreases with increasing density of the bunch.

Consider now the variation range of the value of $\rho(0)$ for $n_b/n_0 > b_0$. From expressions (46) and (47) one has $-\rho_{\max}(0) \leq \rho(0) \leq 0$, where $\rho_{\max}(0)$ is now a function of the density of beam

$$\rho_{\max}(0) = \frac{\gamma_b^2}{1 - \gamma_b^2 \sin^2 \alpha} \left[\cos \alpha \sqrt{G^2 - (\gamma_b^2 - \sin^2 \alpha)} - \beta_b G \right], \quad (52)$$

where

$$G = \cos \alpha + \frac{\sqrt{1 - \gamma_b^2 \sin^2 \alpha} \left(\gamma_b \cos \alpha - \sqrt{1 - \gamma_b^2 \sin^2 \alpha} \right)}{(n_b/n_0) \cos \alpha}. \quad (53)$$

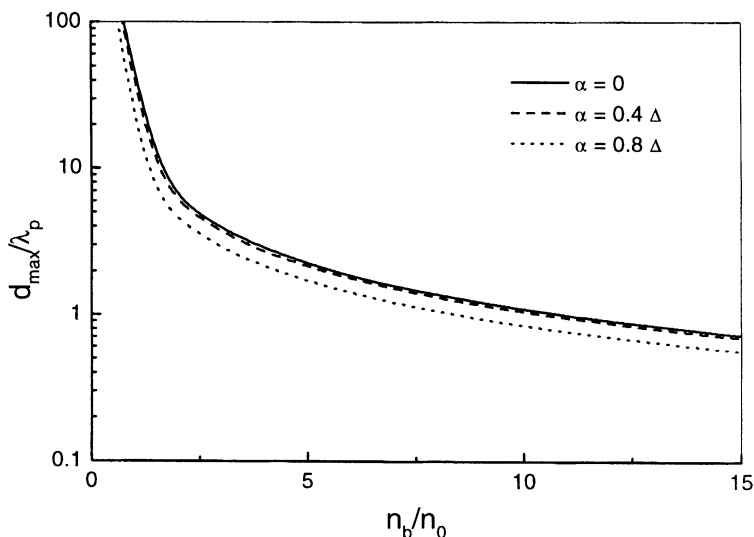


FIGURE 7 The dependence of the d_{\max}/λ_p on n_b/n_0 ($n_b/n_0 \geq b_0$) for $\gamma_b = 60$, $\alpha = 0$ (solid line), $\alpha = 0.4\Delta$ (dashed line) and $\alpha = 0.8\Delta$ (dotted line).

One can find the values of electric fields at the rear boundary of the bunch ($\xi = 0$) for $d = d_{\max}$ from expressions (13), (14), and (18), if one makes a substitution $\rho = -\rho_{\max}(0)$ in the latter, where $\rho_{\max}(0)$ is determined by the expression (52). As a result we have for the value of $\Phi(0)$

$$\Phi(0) = -\sqrt{2} \left\{ \gamma_b \cos \alpha \sqrt{1 - \gamma_b^2 \sin^2 \alpha} - \gamma_b^2 \left(G \cos \alpha - \beta_b \sqrt{G^2 - (\gamma_b^{-2} - \sin^2 \alpha)} \right) \right\}^{1/2}. \quad (54)$$

The maximum values of the longitudinal and transverse electric fields inside the bunch are determined by expressions (36), (38) and (54).

VI. THE TRANSFORMER RATIO

In this section we shall consider the transformer ratio R that is defined as the ratio of maximum value of longitudinal wake field to that

of decelerating field inside the bunch $R = E_{z\max}^{(\text{out})}/E_{z\max}^{(\text{in})}$.⁴⁻⁹ Since the periodic wake field is excited only in case of $\alpha < \Delta$, below we shall consider the transformer ratio precisely for these values of angle α .

We would briefly remind the results obtained for R in the absence of magnetic field. As follows from the results of Refs. [4-9], the transformer ratio monotonically increases with density of bunch, the maximum value $R_{\max} \simeq \sqrt{2(\gamma_b - 1)}$ being achieved at $n_b/n_0 \simeq 1/(2 + 1/\gamma_b)$. In the linear case ($n_b/n_0 \ll 1$) the function R assumes the value $R = 2$. When $n_b/n_0 > 1/(2 + 1/\gamma_b)$, the transformer ratio is monotonically decreasing and tends to unity, $R \simeq 1$, at very large values of bunch density. So, in this case for excitation of wake waves the optimal is a bunch with length $d \leq d_{\max} \equiv \lambda_{\text{in}}/2$ and density $1/(2 + 1/\gamma_b) \leq n_b/n_0 \leq 1/(1 + \beta_b)$, for which the amplitude of wake waves and the transformer ratio take maximum values.

Consider now the influence of external strong magnetic field on the transformer ratio. In case of bunches of low density, when $n_b/n_0 \leq b_0$, the maximum value of longitudinal wake field is determined from Eq. (43), where $d = (n + 1/2)\lambda_{\text{in}}$, ($n = 0, 1, 2, \dots$) and $\rho(0) = -\rho_0$. One can determine the maximum decelerating field inside the bunch from expressions (35) and (36). For very small values of the density of bunch ($n_b/n_0 \ll b_1$) and $n = 0$ we find from expressions (35), (36) and (43):

$$R_0(\alpha) \simeq \frac{2 \cos^2 \alpha + \pi \beta_b^2 \gamma_b^2 \sin^2 \alpha}{\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)[1 + (\gamma_b^2 - 2) \sin^2 \alpha] - 2\beta_b^2 \gamma_b^2 \sin^2 \alpha \arcsin\left(\sqrt{(1 - \gamma_b^2 \sin^2 \alpha)/2 \cos^2 \alpha}\right)}}, \quad (55)$$

that for two limiting values of the angle α takes on the following form:

$$R_0(\alpha \rightarrow 0) \simeq 2; \quad R_0(\alpha \rightarrow \Delta) \simeq \frac{3(2 + \pi)}{2\sqrt{2}} \left(\frac{1 - \gamma_b^{-2}}{1 - \gamma_b^2 \sin^2 \alpha} \right)^{3/2}. \quad (56)$$

The maximum wake field as well as the transformer ratio R grow monotonically with increasing density of bunch, the maximum value of the field being achieved at $n_b/n_0 = b_0$ and being determined by Eq. (49).

As was mentioned above, at $n_b/n_0 > b_0$ the thickness of the bunch is restricted by the value d_{\max} (see Eq. (51)). If the density of bunch is in the

interval $b_0 < n_b/n_0 < b_c$, then one can obtain the maximum value of decelerating field from expressions (35) and (36), where the expression $(d - \tilde{\xi})/\lambda_p$ itself is determined by Eq. (20) with $\rho = -\tilde{\rho}$. For large densities of the bunch $n_b/n_0 > b_c$ the longitudinal field inside the bunch monotonically decrease from a definite value at the first boundary to zero at the second boundary of the bunch. Then the maximum value of decelerating field inside the bunch is equal to the value of field at the first boundary (when $\xi = 0$) and is determined by expression (37). In the range of $n_b/n_0 > b_0$ as a whole, the maximum value of longitudinal wake field is $|\mathcal{E}_{1z}|$ (see Eq. (43)), and, hence, in the last two cases (Eqs. (37) and (43)) instead of ρ_0 and d one is to put their maximum allowable values (see Eqs. (51)–(53)).

In the limit of large densities of the bunch ($n_b/n_0 \gg b_c$) the following expression is obtained from Eq. (51):

$$d_{\max} \simeq \frac{\lambda_p}{(n_b/n_0)} \sqrt{\frac{2}{\cos^2 \alpha}} \sqrt{1 - \gamma_b^2 \sin^2 \alpha} \left(\gamma_b \cos \alpha - \sqrt{1 - \gamma_b^2 \sin^2 \alpha} \right), \quad (57)$$

that enables one to find asymptotic form of the transformer ratio for large densities of the bunch:

$$R_{\infty}(\alpha) \simeq \frac{1 + (\gamma_b^2 - 2) \sin^2 \alpha}{1 - \gamma_b^2 \sin^2 \alpha}. \quad (58)$$

As follows from expressions (55), (56) and (58), in the presence of strong external magnetic field in plasma, the transformer ratio increases and for $\alpha \lesssim \Delta$ much exceeds the value of R at $B_0 = 0$.

In Figures 8–10 the dependence of R on the density of bunch (n_b/n_0) (Figures 8 and 9) and on the angle α (Figure 10) is shown for $\gamma_b = 60$. In Figure 8 the dotted, dashed and solid lines correspond to the following values of the angle α respectively: $\alpha = 0, 0.2\Delta$ and 0.4Δ . In Figure 9 the dotted and solid lines have been plotted for $\alpha = 0.6\Delta$ and 0.8Δ respectively. In Figure 10 the solid line corresponds to $n_b/n_0 = 0.7$, the dashed and dotted lines respectively to $n_b/n_0 = 1$ and 1.3 .

One can assure that the dotted line in Figure 8, for which $\alpha = 0$, coincides with an analogous result obtained in Ref. [9] in the absence of external magnetic field.

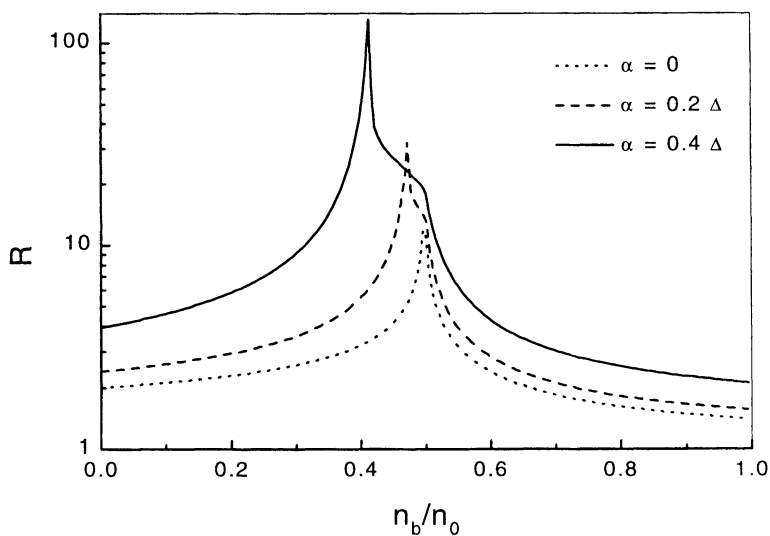


FIGURE 8 The dependence of the transformer ratio R on n_b/n_0 for $\gamma_b = 60$, $\alpha = 0$ (solid line), $\alpha = 0.2\Delta$ (dashed line) and $\alpha = 0.4\Delta$ (dotted line).

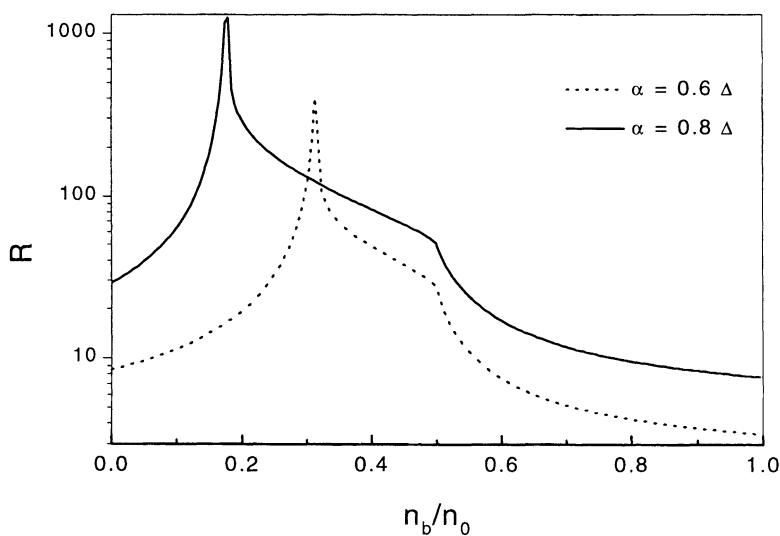


FIGURE 9 The dependence of the transformer ratio R on n_b/n_0 for $\gamma_b = 60$, $\alpha = 0.6\Delta$ (dotted line) and $\alpha = 0.8\Delta$ (solid line).

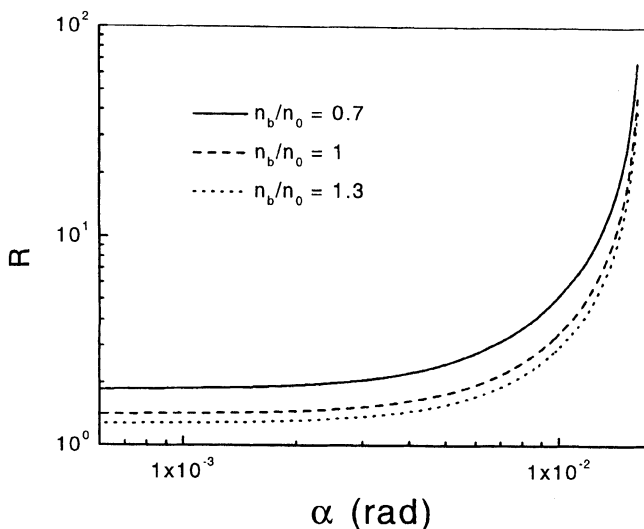


FIGURE 10 The dependence of the transformer ratio R on anisotropy angle α for $\gamma_b = 60$, $n_b/n_0 = 0.7$ (solid line), $n_b/n_0 = 1$ (dashed line) and $n_b/n_0 = 1.3$ (dotted line).

Thus, it follows from the above analysis and from Figures 8–10 that the presence of strong external magnetic field increases the amplitude of wake waves excited by the one-dimensional bunch as well as the transformer ratio. This effect is most strongly manifested in the case when the angle of magnetic field orientation is close to Δ ($\alpha \lesssim \Delta$). Note also that the transformer ratio is maximum at $n_b/n_0 \simeq b_0$, and with increasing α the position of this value is shifted to lower densities of the bunch. Hence, in the presence of strong magnetic field the larger values of transformer ratio may be obtained at much lower densities of the bunch than in the case of $B_0 = 0$.

VII. CONCLUSIONS

In conclusion we shall give a brief summary of the results and discuss possible restrictions on the validity of conclusions.

The behavior of one-dimensional electromagnetic waves excited in plasma in the presence of external strong magnetic field, strongly depends on the orientation of bunch motion with respect to the direction of magnetic field. At large values of angle α ($\alpha \geq \Delta = \arcsin(1/\gamma_b)$)

the density and momentum of plasma electrons, as well as the excited electromagnetic fields vary in the space monotonically. Here the electromagnetic field increases linearly with the distance from the bunch. Obviously, this result is valid only in the one-dimensional model and in case of an allowance for the transverse size of the bunch (i.e., in the three-dimensional case) the fields will tend to zero at large distances. In Ref. [16] an excitation of three-dimensional linear wake waves was considered in the magnetized plasma. It was shown here that at large distances from the driving bunch the fields nevertheless tend to zero although slowly (by the power-law dependence). So, the linear increase of one-dimensional wake fields behind the bunch is limited by the distance that is proportional to the cross-size of the bunch.

The allowance for the transverse size of the bunch results also in the following additional restrictions [17]: $\gamma_b^2 \lambda_p^2 \ll a^2$ and $r^2 \ll a^2$, where a is the transverse size of the bunch and r is the transverse coordinate.

At small values of the angle α ($\alpha < \Delta$) a periodic wake fields is excited behind the bunch, the amplitude of which increases when $\alpha \rightarrow \Delta$. In this case the transformer ratio also increases, the position of its maximum being displaced to lower densities of the bunch.

It follows from the aforesaid that optimal conditions for the excitation of wake waves are $d \lesssim \lambda_{in}/2$ and $n_b/n_0 \simeq b_0(\alpha)$, where $b_0(\alpha)$ is determined by expression (48).

Another feature due to the presence of the magnetic field is the excitation of transverse electric and magnetic fields, that are absent in the one-dimensional case at $B_0 = 0$.⁴⁻¹⁰

The allowance for thermal motion of plasma electrons leads to inessential corrections to wake wave characteristics at temperatures practically achievable under laboratory conditions [18].

Acknowledgment

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APPENDIX

To investigate the set of Eqs. (6)–(11) it is convenient to exclude the electric field and obtain an equation containing the components of dimensionless momentum:

$$\begin{aligned} & \frac{d^2}{d\xi^2} \left[\cos \alpha \sqrt{1 + \rho^2} - \beta_b (\rho_x \sin \alpha + \rho_z \cos \alpha) \right] \\ & + \frac{1}{\lambda_p^2} \frac{\beta_b^2 \gamma_b^2 \rho_x \sin \alpha - \rho_z \cos \alpha}{\beta_b \sqrt{1 + \rho^2} - \rho_z} = \frac{\cos \alpha}{\lambda_p^2} \frac{n_b}{n_0}. \end{aligned} \quad (A1)$$

Then the component ρ_y is eliminated from Eqs. (9) and (11). The following equation is obtained:

$$\begin{aligned} & \frac{d}{d\xi} (\rho_x \sin \alpha + \rho_z \cos \alpha) = \\ & \frac{e}{mc^2 \beta_b} \cos \alpha \frac{\left(\beta_b \sqrt{1 + \rho^2} \operatorname{tg} \alpha + \rho_x - \rho_z \operatorname{tg} \alpha \right) E_x + \rho_y E_y + \beta_b \sqrt{1 + \rho^2} E_z}{\beta_b \sqrt{1 + \rho^2} - \rho_z}. \end{aligned} \quad (A2)$$

In the limit of strong magnetic field ($\omega_B \gg \omega_p$ or $\lambda_B \ll \lambda_p$) and in the zeroth-order approximation in λ_B from Eqs. (A1) and (A2) we obtain:

$$\begin{aligned} \frac{d^2}{d\xi^2} (\cos \alpha \sqrt{1 + \rho^2} - \beta_b \rho) \\ + \frac{\gamma_b^2 \sin^2 \alpha - 1}{\lambda_p^2} \frac{\rho}{\beta_b \sqrt{1 + \rho^2} - \rho \cos \alpha} = \frac{\cos \alpha}{\lambda_p^2} \frac{n_b}{n_0}, \end{aligned} \quad (\text{A3})$$

$$\frac{\beta_b \sqrt{1 + \rho^2} - \rho \cos \alpha}{\sqrt{1 + \rho^2}} \frac{d\rho}{d\xi} = \frac{e}{mc^2} (E_x \sin \alpha + E_z \cos \alpha). \quad (\text{A4})$$

To express the components of electric field E_x and E_z through the dimensionless momentum of electrons ρ , one is to obtain the second equation that connects the components of the field. One can derive it from Eqs. (4) and (5) by eliminating the momentum of electrons. As a result one has

$$\frac{d}{d\xi} (E_x \cos \alpha + \beta_b^2 \gamma_b^2 E_z \sin \alpha) = -\frac{mc\omega_p}{e} \beta_b^2 \gamma_b^2 \frac{\sin \alpha}{\lambda_p} \frac{n_b}{n_0}. \quad (\text{A5})$$